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Semiclassical Study of the Quantum Back-Reaction Problem in Scalar QED

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The quantum back-reaction problem in scalar QED is studied, in the semiclassical approximation, for nonstationary external electric fields. Using a model proposed in [Cooper, F., Mottola, E. (1989). Quantum back reaction in scalar QED as an initial-value problem. *Physical Review* **D40**(2), 456–464], based in a Hartree-type equation, we calculate, until order $\hbar$, the induced current density and the induced electromagnetic field. Once we have computed the induced electromagnetic field, we obtain, until order $\hbar$, the effective Lagrangian and the average density of produced pairs taking into account the back-reaction.

KEY WORDS: quantum electrodynamics; back-reaction; semiclassical approach.
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Notation

$A^{\mu}_{\text{ext}} := (\phi_{\text{ext}}, \mathbf{A}_{\text{ext}})$	External electromagnetic potential
$A^{\mu} := (\phi, \mathbf{A})$	Induced electromagnetic potential
$D^{\mu} := (D^0, \mathbf{D}) := \partial^{\mu} - \frac{ie}{\hbar c}\left(A^{\mu}_{\text{ext}} + A^{\mu}\right)$	Covariant derivate, where $e > 0$
$\mathbf{p}_{\mathbf{k}}$	$\frac{2\pi\hbar}{L}\mathbf{k}$
$\lvert\mathbf{p}\rvert$	Norm of the vector $\mathbf{p}$
$g(t^{\pm})$	$\lim_{\epsilon\to 0} g(t \pm \lvert\epsilon\rvert)$
$\mathcal{C}^r$	Space of functions with continuous derivate at order r
$\alpha := \frac{e^2}{\hbar c}$	Fine structure constant.

1. INTRODUCTION

In some papers, the problem of pair creation in the presence of a classical electromagnetic background has been investigated. Different situations are

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studied: Pair production for analytic and nonanalytic uniform electric fields (Fulling, 1989; Gavrilov and Gitman, 1995; Marinov and Popov, 1977; Nikishov, 1970; Popov, 1972), pair production at time t, for uniform and no uniform electric fields (Dolby and Gull, 2002; Grib *et al.*, 1994; Haro, 2004). In these papers, the authors study the pair-production phenomenon in the so-called "one-loop approximation," that is, they do not take into account either the effect of the produced particles on the electromagnetic field (the back-reaction) nor the subsequent effect of the induced electromagnetic field on the pair-production phenomenon.

In order to study the back-reaction problem, Fulling (1989) proposed a model based on a Hartree-type equation and, with this model, they obtained some numerical results (see Cooper *et al.*, 1991). The aim of the present paper was to apply the background developed in (Haro, 2003) for the model proposed in (Fulling, 1989). In fact, using the model proposed in (Fulling, 1989) in the Schrödinger picture, we obtain, after charge renormalization, semiclassical formulae for scalar QED, that generalize the results obtained in Grib *et al.* (1994), Haro (2003), and Mostepanenko (1979), that is, we obtain, until order $\hbar$, the induced current density and the induced electromagnetic field. Once we have obtained the induced electromagnetic field, we generalize, until order $\hbar$, the formula of the effective Lagrangian in the one-loop approximation and the formula of the average density of produced pairs in the one-loop approximation.

The paper is organized as follows. In Section 2, we present the model proposed in (Fulling, 1989), in the Schrödinger picture. With this model, we compute, until order $\hbar$, the induced electric field and the induced current density in the case of an uniform external electric field. In Section 3, we calculate, in the case of an uniform electric field, the effective Lagrangian density taking into consideration the back-reaction. Using the relativistic invariance requirement (see Grib *et al.*, 1994), we generalize the obtained formula in the case where exists an external magnetic field perpendicular to the external electric field. In section 4, we compute the energy density and the average density of produced pairs at time t, taking into account the back-reaction. Finally in Section 5, using the W.K.B. approximation, we study the back-reaction problem for discontinuous external fields and, we show the main differences with the continuous case.

2. THE MODEL IN THE SCHRÖDINGER PICTURE

Let $\hat{\psi}$ and $\hat{\pi}^{\dagger}$ be the canonically conjugated operators that define the scalar massive field. With this operators and the normal ordering operator ::, we can define, in the Schrödinger picture, the current density operator

$\hat{j}^\mu(t,\mathbf{x}) := (c\hat{\rho}(t,\mathbf{x}), \hat{\mathbf{j}}(t,\mathbf{x}))$, where

$$\hat{\rho} = -\frac{ie}{\hbar} : \hat{\pi}\hat{\psi}^\dagger - \hat{\pi}^\dagger\hat{\psi} : \quad \hat{\mathbf{j}} = -i\hbar ec : \hat{\psi}^\dagger\mathbf{D}\hat{\psi} - \hat{\psi}\mathbf{D}^*\hat{\psi}^\dagger :, \tag{1}$$

and the Hamiltonian operator (Greiner *et al.*, 1985; Weinberg, 1995)

$$\hat{H}(t) = \int d^3x \left[\frac{1}{\hbar^2} : \hat{\pi}\hat{\pi}^\dagger : + : i\hbar c\mathbf{D}\hat{\psi} \cdot (i\hbar c\mathbf{D})^*\hat{\psi}^\dagger : + m^2c^4 : \hat{\psi}\hat{\psi}^\dagger : + l(\phi_{\text{ext}} + \phi)\hat{\rho} \right]. \tag{2}$$

Now let $A^\mu(t,\mathbf{x})$ be the renormalized induced electromagnetic potential, defined by

$$A^\mu(t,\mathbf{x}) := B^\mu(t,\mathbf{x}) + C\alpha(A^\mu_{\text{ext}}(t,\mathbf{x}) + A^\mu(t,\mathbf{x})), \tag{3}$$

where C is a divergent dimensionless constant that we calculate explicitly in the next subsection and, $B^\mu(\mathbf{x},t)$ is the "divergent" electromagnetic potential induced by the divergent current density.

Once we have introduced the renormalized electromagnetic field, the dynamical equations in the Schrödinger picture are (see Fulling, 1989; Cooper *et al.*, 1991)

$$\begin{cases} i\hbar\partial_t\mathcal{T} & = \hat{H}(t)\mathcal{T} \\ \mathcal{T}(-\infty) & = Id \\ \frac{1}{c^2}\ddot{B}^\mu(t,\mathbf{x}) - \Delta B^\mu(t,\mathbf{x}) & = \frac{4\pi}{c}\langle 0|\mathcal{T}^\dagger(t)\hat{j}^\mu(t,\mathbf{x})\mathcal{T}(t)|0\rangle \\ B^\mu(-\infty,\mathbf{x}) & = (0,\mathbf{0}) \\ \dot{B}^\mu(-\infty,\mathbf{x}) & = (0,\mathbf{0}). \end{cases} \tag{4}$$

Finally, we can define the renormalized current density

$$j^\mu(t,\mathbf{x}) := \langle 0|\mathcal{T}^\dagger(t)\hat{j}^\mu(t,\mathbf{x})\mathcal{T}(t)|0\rangle + C\alpha(j^\mu_{\text{ext}}(t,\mathbf{x}) + j^\mu(t,\mathbf{x})), \tag{5}$$

where $|0\rangle$ is the vacuum state at time $-\infty$, that coincides with the free-vacuum state, because we suppose that the interaction is adiabatically switched on.

2.1. A Semiclassically Solvable Example

Here we consider the case $A^\mu_{\text{ext}}(t,\mathbf{x}) = (0, \mathbf{f}_{\text{ext}}(t))$. Following the method described in Haro (2003), we introduce the creation and annihilation operators at time t in the Schrödinger picture:

$$\hat{a}_{\mathbf{k}}(t) = \frac{1}{2\sqrt{\in_{\mathbf{k}}(t)}}\left[\left(i\hat{P}_{\kappa} + \omega_{\mathbf{k}}(t)\hat{Q}_{\mathbf{k}}\right) + i\left(i\hat{\hat{P}}_{\mathbf{k}} + \omega_{\mathbf{k}}(t)\hat{\hat{Q}}_{\mathbf{k}}\right)\right]$$

$$\hat{b}^{\dagger}_{-\mathbf{k}}(t) = \frac{1}{2\sqrt{\in_{\mathbf{k}}(t)}}\left[\left(-i\hat{P}_{\mathbf{k}} + \omega_{\mathbf{k}}(t)\hat{Q}_{\mathbf{k}}\right) + i\left(-i\hat{\hat{P}}_{\mathbf{k}} + \omega_{\mathbf{k}}(t)\hat{\hat{Q}}_{\mathbf{k}}\right)\right],$$

where $\epsilon_{\mathbf{k}}(t) = \sqrt{|c\mathbf{p}_{\mathbf{k}} + e\,(\mathbf{f}_{\text{ext}}(t) + \mathbf{f}(t))|^2 + m^2c^4}$ is the energy of a particle at time t, $\omega_{\mathbf{k}}(t) \equiv \frac{\epsilon_{\mathbf{k}}(t)}{\hbar}$ is the frequency and, $(\hat{P}_{\mathbf{k}}, \hat{Q}_{\mathbf{k}})$, $(\hat{\hat{P}}_{\mathbf{k}}, \hat{\hat{Q}}_{\mathbf{k}})$ are the canonically conjugated operators that appear in the decomposition of the energy operator in an infinite set of harmonic oscillators (see Greiner *et al.*, 1985; Haro, 2003).

Then, using these operators we can write the canonically conjugated operators $\hat{\psi}$ and $\hat{\pi}$ in the following form:

$$\hat{\psi}(t, \boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^3} \frac{1}{\sqrt{2\epsilon_{\boldsymbol{k}}(t)L^3}}\left(\hat{a}_{\boldsymbol{k}}(t) + \hat{b}^{\dagger}_{-\boldsymbol{k}}(t)\right)e^{\frac{2\pi i}{L}\boldsymbol{kx}}$$

$$\hat{\pi}(t, \boldsymbol{x}) = -i\hbar\sum_{\boldsymbol{k}\in\mathbb{Z}^3}\sqrt{\frac{\epsilon_{\boldsymbol{k}}(t)}{2L^3}}\left(\hat{a}_{\boldsymbol{k}}(t) - \hat{b}^{\dagger}_{-\boldsymbol{k}}(t)\right)e^{\frac{2\pi i}{L}\boldsymbol{kx}}.$$

It is easy to check that the energy and the current density operators have the following decomposition:

$$\hat{H}(t) = \sum_{\mathbf{k}\in\mathbb{Z}^3}\epsilon_{\mathbf{k}}(t)\left(\hat{a}^{\dagger}_{\mathbf{k}}(t)\hat{a}_{\mathbf{k}}(t) + \hat{b}^{\dagger}_{-\mathbf{k}}(t)\hat{b}_{-\mathbf{k}}(t)\right),$$

$$\hat{\rho}(t) := \frac{1}{L^3}\int d^3x\,\hat{\rho}(t, \mathbf{x}) = -\frac{e}{L^3}\sum_{\mathbf{k}\in\mathbb{Z}^3}\left(\hat{a}^{\dagger}_{\mathbf{k}}(t)\hat{a}_{\mathbf{k}}(t) - \hat{b}^{\dagger}_{-\mathbf{k}}(t)\hat{b}_{-\mathbf{k}}(t)\right),$$

$$\begin{aligned}\hat{\mathbf{j}}(t) &:= \frac{1}{L^3}\int d^3x\,\hat{\mathbf{j}}(t, \mathbf{x})\\ &= -\frac{ec}{L^3}\sum_{\mathbf{k}\in\mathbb{Z}^3}\frac{c\mathbf{p}_{\mathbf{k}} + e(\mathbf{f}_{\text{ext}}(t) + \mathbf{f}(t))}{\epsilon_{\mathbf{k}}(t)}\left(\hat{a}^{\dagger}_{\mathbf{k}}(t)\hat{a}_{\mathbf{k}}(t) + \hat{b}^{\dagger}_{-\mathbf{k}}(t)\hat{b}_{-\mathbf{k}}(t)\right.\\ &\quad\left. + \hat{b}_{-\mathbf{k}}(t)\hat{a}_{\mathbf{k}}(t) + \hat{b}^{\dagger}_{-\mathbf{k}}(t)\hat{a}^{\dagger}_{\mathbf{k}}(t)\right).\end{aligned}$$

Note that, the renormalized potential has the following form $A^{\mu}(t, \mathbf{x}) = (0, \mathbf{f}(t))$, with $\mathbf{f}(t) := \mathbf{g}(t) + C\alpha(\mathbf{f}_{\text{ext}}(t) + \mathbf{f}(t))$, where $\mathbf{g}(t)$ is the "divergent"-induced vector potential.

Then, in this case, the dynamical equations can be written in the following form:

$$i\hbar\partial_t \mathcal{T} = \hat{H}(t)\mathcal{T}; \qquad \mathbf{g}(t) = 4\pi c \int_{-\infty}^{t} d\tau \int_{-\infty}^{\tau} ds \langle 0|\mathcal{T}^{\dagger}(s)\hat{\mathbf{j}}(s)\mathcal{T}(s)|0\rangle. \quad (6)$$

Therefore, the renormalized vector current density is $\mathbf{j}(t) := \langle 0|\mathcal{T}^{\dagger}(t)\hat{\mathbf{j}}(t)\mathcal{T}(t)|0\rangle + C\alpha(\mathbf{j}_{\text{ext}}(t) + \mathbf{j}(t))$.

In order to obtain the solutions of problem (6), first, using the semiclassical approximation, we solve the following problem

$$\begin{cases} i\hbar\partial_t \mathcal{T} & = \hat{H}(t)\mathcal{T} \\ \mathcal{T}(-\infty) = Id, \end{cases} \quad (7)$$

that is, in the semiclassical approximation, we compute $\mathcal{T}(t)|0\rangle$. And once we have it, we calculate $\mathbf{f}(t)$ and $\mathbf{j}(t)$.

Note that, the solution of the problem (7) has the following form:

$$\mathcal{T}(t)|0\rangle = \prod_{\mathbf{k}\in\mathbb{Z}^3} T_{\mathbf{k}}(t)\phi_{\mathbf{k}}^{0,0}(-\infty), \quad (8)$$

where $T_{\mathbf{k}}(t)\phi_{\mathbf{k}}^{0,0}(-\infty)$ is the solution of the problem

$$\begin{cases} i\hbar\partial_t \phi(t) = \hat{H}_{\mathbf{k}}(t)\phi(t) \\ \phi(-\infty) \ = \phi_{\mathbf{k}}^{0,0}(-\infty), \end{cases} \quad (9)$$

with

$$\phi_{\mathbf{k}}^{0,0}(t) \quad = \sqrt{\frac{\omega_{\mathbf{k}}(t)}{\pi\hbar}} \exp\left(-\frac{\omega_{\mathbf{k}}(t)}{2\hbar}(Q_{\mathbf{k}}^2 + \bar{Q}_{\mathbf{k}}^2)\right) \quad \text{and}$$

$$\hat{H}_{\mathbf{k}}(t) = \epsilon_{\mathbf{k}}(t)\left(\hat{a}_{\mathbf{k}}^{\dagger}(t)\hat{a}_{\mathbf{k}}(t) + \hat{b}_{-\mathbf{k}}^{\dagger}(t)\hat{b}_{-\mathbf{k}}(t)\right).$$

We write the solution of the problem (9) in powers of $\hbar$, (see Haro, 2003):

$$T_{\mathbf{k}}(t)\phi_{\mathbf{k}}^{0,0}(-\infty) = \sum_{j,s\in\mathbb{N}} \hbar^{s+j} A_{s,\mathbf{k}}^{j}(t)\phi_{\mathbf{k}}^{s,s}(t), \quad (10)$$

where

$$\phi_{\mathbf{k}}^{s,s}(t) := \frac{\left(\hat{a}_{\mathbf{k}}^{\dagger}(t)\right)^s \left(\hat{b}_{-\mathbf{k}}^{\dagger}(t)\right)^s}{s!}\phi_{\mathbf{k}}^{0,0}(t).$$

The functions $A^j_{s,\mathbf{k}}(t)$ satisfy the system:

$$A^0_{0,\mathbf{k}}(t) = 1; \qquad A^0_{1,\mathbf{k}}(t) = -i\,\frac{\dot{\epsilon}_{\mathbf{k}}(t)}{4\epsilon^2_{\mathbf{k}}(t)}.$$

If $s = 0$

$$\dot{A}^j_{0,\mathbf{k}}(t) = -2i\epsilon_{\mathbf{k}}(t)A^0_{1,\mathbf{k}}(t)A^{j-1}_{1,\mathbf{k}}(t) = 0, \quad for \quad j > 0.$$

If $s > 0$

$$A^0_{s,\mathbf{k}}(t) = A^0_{1,\mathbf{k}}(t)A^0_{s-1,\mathbf{k}}(t)$$

$$sA^1_{s,\mathbf{k}}(t) = \frac{i}{2\epsilon_{\mathbf{k}}(t)}\dot{A}^0_{s,\mathbf{k}}(t) + sA^0_{1,\mathbf{k}}(t)A^1_{s-1,\mathbf{k}}(t)$$

$$sA^j_{s,\mathbf{k}}(t) = \frac{i}{2\epsilon_{\mathbf{k}}(t)}\dot{A}^{j-1}_{s,\mathbf{k}}(t) - A^0_{1,\mathbf{k}}(t)\left((s+1)A^{j-2}_{s+1,\mathbf{k}}(t) - sA^j_{s-1,\mathbf{k}}(t)\right), \quad \text{for} \quad j > 1.$$

Now, in order to compute $\langle 0|\mathcal{T}^\dagger(t)\hat{\mathbf{j}}(t)\mathcal{T}(t)|0\rangle$ until terms at order $\hbar$, we use (10) until order $\hbar^4$. In this approximation, a cumbersome but elementary computation provides the following result

$$\begin{aligned}\langle 0|\mathcal{T}^\dagger(t)\hat{\mathbf{j}}(t)\mathcal{T}(t)|0\rangle = & -\frac{\alpha}{4\pi c}C(\ddot{\mathbf{f}}_{\text{ext}}(t) + \ddot{\mathbf{f}}(t)) + \frac{\alpha\left(\frac{\hbar}{mc^2}\right)^2}{480\pi^2 c}\frac{d^4}{dt^4}(\mathbf{f}_{\text{ext}}(t) + \mathbf{f}(t)) \\ & -\frac{7\alpha^2\left(\frac{\hbar}{mc}\right)^3}{1440\pi^2c^3mc^2}\frac{d}{dt}\left(\left(\dot{\mathbf{f}}_{\text{ext}}(t) + \dot{\mathbf{f}}(t)\right)|\dot{\mathbf{f}}_{\text{ext}}(t) + \dot{\mathbf{f}}(t)|^2\right),\end{aligned} \tag{11}$$

where we have introduced the dimensionless divergent constant $C = \frac{1}{6\pi}\int_0^\infty \frac{x^4}{(x^2+1)^{5/2}}dx$ (see Mostepanenko, 1979).

Consequently, in this approximation, using the induced electric field $\mathbf{E}(t) := -\frac{1}{c}\dot{\mathbf{f}}(t)$, and the external electric field $\mathbf{E}_{\text{ext}}(t) := -\frac{1}{c}\dot{\mathbf{f}}_{\text{ext}}(t)$, we have

$$\mathbf{E}(t) = \frac{\alpha\left(\frac{\hbar}{mc^2}\right)^2}{120\pi}\left(\ddot{\mathbf{E}}_{\text{ext}}(t) + \ddot{\mathbf{E}}(t)\right) - \frac{7\alpha^2\left(\frac{\hbar}{mc}\right)^3}{360\pi mc^2}\left(\mathbf{E}_{\text{ext}}(t) + \mathbf{E}(t)\right)|\mathbf{E}_{\text{ext}}(t) + \mathbf{E}(t)|^2 \tag{12}$$

$$\mathbf{j}(t) = -\frac{\alpha\left(\frac{\hbar}{mc^2}\right)^2}{480\pi^2}(\ddot{\mathbf{E}}_{\text{ext}}(t)+\ddot{\mathbf{E}}(t)) + \frac{7\alpha^2\left(\frac{\hbar}{mc}\right)^3}{1440\pi^2 mc^2} \times \frac{d}{dt}\left((\mathbf{E}_{\text{ext}}(t)+\mathbf{E}(t))\,|\mathbf{E}_{\text{ext}}(t)+\mathbf{E}(t)|^2\right). \tag{13}$$

Note that, we have approximated $\mathbf{E}(t)$ and $\mathbf{j}(t)$ until terms at order $\hbar$, because $\alpha\hbar^2 \sim \hbar$ and $\alpha^2\hbar^3 \sim \hbar$. Then, we can conclude that, until order $\hbar$, the induced electric field and the induced current density are

$$\mathbf{E}(t) = \frac{\alpha\left(\frac{\hbar}{mc^2}\right)^2}{120\pi}\ddot{\mathbf{E}}_{\text{ext}}(t) - \frac{7\alpha^2\left(\frac{\hbar}{mc}\right)^3}{360\pi mc^2}\mathbf{E}_{\text{ext}}(t)|\mathbf{E}_{\text{ext}}(t)|^2 \tag{14}$$

$$\mathbf{j}(t) = -\frac{\alpha\left(\frac{\hbar}{mc^2}\right)^2}{480\pi^2}\ddot{\mathbf{E}}_{\text{ext}}(t) + \frac{7\alpha^2\left(\frac{\hbar}{mc}\right)^3}{1440\pi^2 mc^2}\frac{d}{dt}\left(\mathbf{E}_{\text{ext}}(t)|\mathbf{E}_{\text{ext}}(t)|^2\right). \tag{15}$$

We also note that, this induced current density coincides with formula (6.52), obtained in Grib *et al.* (1994) using the Heisenberg picture.

3. THE EFFECTIVE LAGRANGIAN DENSITY

The effective Lagrangian density in the case $A^{\mu}_{\text{ext}}(t, \mathbf{x}) = (0, \mathbf{f}_{\text{ext}}(t))$ is defined by

$$\mathcal{L}_{\text{eff}}(t) := \frac{1}{L^3}\frac{d}{dt}\left(\frac{\hbar}{i}\log\left(\langle 0_t|\mathcal{T}(t)|0\rangle\right)\right), \tag{16}$$

where $|0_t\rangle := \prod_{\mathbf{k}\in\mathbb{Z}^3}\phi^{0,0}_{\mathbf{k}}(t)$ is the vacuum state at time t and, $\mathcal{T}$ is the solution of the problem (7). Then, using (10) until terms at order $\hbar^4$, we obtain after charge renormalization (see Schwinger, 1951), the following real part of the effective Lagrangian density

$$\mathrm{Re}\left(\mathcal{L}_{\text{eff}}(t)\right) = \frac{1}{8\pi}|\mathbf{E}_{\text{ext}}(t)+\mathbf{E}(t)|^2 + \frac{7\alpha^2\left(\frac{\hbar}{mc}\right)^3}{5760\pi^2 mc^2}|\mathbf{E}_{\text{ext}}(t)+\mathbf{E}(t)|^4 - \frac{\alpha\left(\frac{\hbar}{mc^2}\right)^2}{1920\pi^2}\left(\frac{d^2}{dt^2}|\mathbf{E}_{\text{ext}}(t)+\mathbf{E}(t)|^2 - 2|\dot{\mathbf{E}}_{\text{ext}}(t)+\dot{\mathbf{E}}(t)|^2\right). \tag{17}$$

Note that, in this effective Lagrangian density appears a total derivate, so we can define the equivalent effective Lagrangian density

$$\mathrm{Re}\left(\widetilde{\mathcal{L}}_{\mathrm{eff}}(t)\right) := \frac{1}{8\pi}|\mathbf{E}_{\mathrm{ext}}(t)+\mathbf{E}(t)|^2 + \frac{\alpha\left(\frac{\hbar}{mc^2}\right)^2}{960\pi^2}|\dot{\mathbf{E}}_{\mathrm{ext}}(t)+\dot{\mathbf{E}}(t)|^2 + \frac{7\alpha^2\left(\frac{\hbar}{mc}\right)^3}{5760\pi^2 mc^2}|\mathbf{E}_{\mathrm{ext}}(t)+\mathbf{E}(t)|^4. \tag{18}$$

Therefore, for $\mathbf{E}(t) \equiv \vec{0}$, we obtain the real part of the effective Lagrangian density in the one-loop approximation

$$\mathrm{Re}\left(\widetilde{\mathcal{L}}_{\mathrm{eff},0}(t)\right) := \frac{1}{8\pi}|\mathbf{E}_{\mathrm{ext}}(t)|^2 + \frac{7\alpha^2\left(\frac{\hbar}{mc}\right)^3}{5760\pi^2 mc^2}|\mathbf{E}_{\mathrm{ext}}(t)|^4 + \frac{\alpha\left(\frac{\hbar}{mc^2}\right)^2}{960\pi^2}|\dot{\mathbf{E}}_{\mathrm{ext}}(t)|^2. \tag{19}$$

Until order $\hbar$, if we consider the back-reaction, the real part of the effective Lagrangian density is

$$\mathrm{Re}\left(\widetilde{\mathcal{L}}_{\mathrm{eff}}(t)\right) = \frac{1}{8\pi}|\mathbf{E}_{\mathrm{ext}}(t)|^2 - \frac{7\alpha^2\left(\frac{\hbar}{mc}\right)^3}{1920\pi^2 mc^2}|\mathbf{E}_{\mathrm{ext}}(t)|^4 - \frac{\alpha\left(\frac{\hbar}{mc^2}\right)^2}{960\pi^2}|\dot{\mathbf{E}}_{\mathrm{ext}}(t)|^2. \tag{20}$$

Remark 3.1. Note that, in the case of the external vector potential $\mathbf{f}_{\mathrm{ext}}(t) = -ct\mathbf{E}$, we have

$$\mathrm{Re}\mathcal{L}_{\mathrm{eff},0} = \frac{1}{8\pi}|\mathbf{E}|^2 + \frac{7\alpha^2\left(\frac{\hbar}{mc}\right)^3}{5760\pi^2 mc^2}|\mathbf{E}|^4. \tag{21}$$

This is the one-loop Euler–Heisenberg formula for spinless particles obtained by Schwinger (1951).

Remark 3.2. Note that, in the one-loop approximation the formula (19) differs on the total derivate

$$-\frac{4\alpha\left(\frac{\hbar}{mc^2}\right)^2}{960\pi^2}\frac{d}{dt}\left(\mathbf{E}_{\mathrm{ext}}(t)\cdot\dot{\mathbf{E}}_{\mathrm{ext}}(t)\right),$$

of the formula (6.71) obtained in Grib *et al.* (1994). For this reason, these two formulae are equivalents.

3.1. The General Case

Here, we will generalize the formulae (19) and (20), for the general case of an electric and magnetic field. We follow the way described in Grib *et al.* (1994), i.e., we use the relativistic invariant requirement. If we consider the electromagnetic tensor $F_{\mu\nu,\text{ext}} = \partial_\mu A_{\nu,\text{ext}} - \partial_\nu A_{\mu,\text{ext}}$, the characteristic polynomial of this matrix provides the following two invariant scalars

$$I_1 := (|\mathbf{E}_{\text{ext}}|^2 - |\mathbf{H}_{\text{ext}}|^2); \qquad I_2 := (\mathbf{E}_{\text{ext}} \cdot \mathbf{H}_{\text{ext}})^2,$$

and with this two invariant scalars, we construct the term

$$\frac{\alpha^2 \left(\frac{\hbar}{mc}\right)^3}{mc^2} \left(A I_1^2 + B I_2\right),$$

where A and B are real numbers. Another fundamental invariant scalar that contains the square of the external electromagnetic field is $I_3 := 16\pi^2 \left(\rho_{\text{ext}}^2 - \frac{1}{c^2}|\mathbf{j}_{\text{ext}}|^2\right)$. In fact, using the Maxwell equations we obtain

$$I_3 = (\nabla \cdot \mathbf{E}_{\text{ext}})^2 - | - \frac{1}{c}\dot{\mathbf{E}}_{\text{ext}} + \nabla \wedge \mathbf{H}_{\text{ext}}|^2 = \partial_\alpha F^{\alpha}_{\nu,\text{ext}} \partial_\mu F^{\mu\nu}_{\text{ext}}.$$

Remark 3.3. Note that, if we do not take into account the quadri-divergences, I_3 is equivalent to the invariant scalar

$$F^{\mu\nu}_{\text{ext}} \Box F_{\mu\nu,\text{ext}} = -2(\mathbf{E}_{\text{ext}} \cdot \Box \mathbf{E}_{\text{ext}} - \mathbf{H}_{\text{ext}} \cdot \Box \mathbf{H}_{\text{ext}}),$$

where $\Box$ is the D'Alambertian.

Then, with this fundamental invariant scalar we construct the term $D\alpha \left(\frac{\hbar}{mc}\right)^2 I_3$, where D is a real number. Now, if we concentrate on the formulae (19) and (3.54) of Schwinger (1951), we deduce that

$$\begin{aligned} \text{Re}\left(\widetilde{\mathcal{L}}_{\text{eff},0}(\mathbf{x}, t)\right) = \; & \frac{1}{8\pi}(|\mathbf{E}_{\text{ext}}|^2 - |\mathbf{H}_{\text{ext}}|^2) - \frac{\alpha^2 \left(\frac{\hbar}{mc}\right)^3}{1440\pi^2 mc^2} \\ & \times \left[\frac{7}{4}(|\mathbf{E}_{\text{ext}}|^2 - |\mathbf{H}_{\text{ext}}|^2)^2 + (\mathbf{E}_{\text{ext}} \cdot \mathbf{H}_{\text{ext}})^2\right] \\ & - \frac{\alpha \left(\frac{\hbar}{mc}\right)^2}{960\pi^2} \left((\nabla \cdot \mathbf{E}_{\text{ext}})^2 - \left|-\frac{1}{c}\dot{\mathbf{E}}_{\text{ext}} + \nabla \wedge \mathbf{H}_{\text{ext}}\right|^2\right), \end{aligned} \tag{22}$$

and, if we concentrate on the formula (20), we reach

$$\begin{aligned} \mathrm{Re}\left(\widetilde{\mathcal{L}}_{\mathrm{eff}}(\mathbf{x},t)\right) = \; & \frac{1}{8\pi}(|\mathbf{E}_{\mathrm{ext}}|^2 - |\mathbf{H}_{\mathrm{ext}}|^2) - \frac{7\alpha^2\left(\frac{\hbar}{mc}\right)^3}{1920\pi^2 mc^2} \\ & \times \left[(|\mathbf{E}_{\mathrm{ext}}|^2 - |\mathbf{H}_{\mathrm{ext}}|^2)^2 + \bar{B}(\mathbf{E}_{\mathrm{ext}}\cdot\mathbf{H}_{\mathrm{ext}})^2\right] \\ & + \frac{\alpha\left(\frac{\hbar}{mc}\right)^2}{960\pi^2}\left((\nabla\cdot\mathbf{E}_{\mathrm{ext}})^2 - \left|-\frac{1}{c}\dot{\mathbf{E}}_{\mathrm{ext}} + \nabla\wedge\mathbf{H}_{\mathrm{ext}}\right|^2\right), \quad (23) \end{aligned}$$

where, in order to obtain the real number $\bar{B}$, we must obtain the fourth order vacuum polarization using the perturbation theory. This calculation is very complicated and will be do in a subsequent paper. Finally, we remark that in the case that the external electric and magnetic fields are perpendicular, the formula (25) becomes

$$\begin{aligned} \mathrm{Re}\left(\widetilde{\mathcal{L}}_{\mathrm{eff}}(\mathbf{x},t)\right) = \; & \frac{1}{8\pi}(|\mathbf{E}_{\mathrm{ext}}|^2 - |\mathbf{H}_{\mathrm{ext}}|^2) - \frac{7\alpha^2\left(\frac{\hbar}{mc}\right)^3}{1920\pi^2 mc^2}(|\mathbf{E}_{\mathrm{ext}}|^2 - |\mathbf{H}_{\mathrm{ext}}|^2)^2 \\ & + \frac{\alpha\left(\frac{\hbar}{mc}\right)^2}{960\pi^2}\left((\nabla\cdot\mathbf{E}_{\mathrm{ext}})^2 - \left|-\frac{1}{c}\dot{\mathbf{E}}_{\mathrm{ext}} + \nabla\wedge\mathbf{H}_{\mathrm{ext}}\right|^2\right). \quad (24) \end{aligned}$$

4. THE DENSITY OF PRODUCED PAIRS AND THE ENERGY DENSITY OF THE VACUUM

In this section, we calculate the average density of produced pairs and the energy density of the vacuum state in the presence of an external vector potential $\mathbf{f}_{\mathrm{ext}}(t)$ taking into account the quantum back-reaction. That is, we calculate

$$\mathcal{E}(t) := \frac{1}{L^3}\langle 0|\mathcal{T}^\dagger(t)\hat{H}(t)\mathcal{T}(t)|0\rangle \quad \text{and} \quad \mathcal{N}(t) := \frac{1}{L^3}\langle 0|\mathcal{T}^\dagger(t)\hat{N}(t)\mathcal{T}(t)|0\rangle,$$

where $\hat{N}(t) := \sum_{\mathbf{k}\in\mathbb{Z}^3}\hat{a}^\dagger_{\mathbf{k}}(t)\hat{a}_{\mathbf{k}}(t)$ is the quantum operator "number of particles at time t".

After charge renormalization, the obtained result until order $\hbar$, in the one-loop approximation is

$$\begin{aligned} \mathcal{E}_0(t) := \; & \frac{1}{8\pi}|\mathbf{E}_{\mathrm{ext}}(t)|^2 + \frac{7\alpha^2\left(\frac{\hbar}{mc}\right)^3}{1920\pi^2 mc^2}|\mathbf{E}_{\mathrm{ext}}(t)|^4 \\ & + \frac{\alpha\left(\frac{\hbar}{mc^2}\right)^2}{960\pi^2}\left(|\dot{\mathbf{E}}_{\mathrm{ext}}(t)|^2 - 2\mathbf{E}_{\mathrm{ext}}(t)\cdot\ddot{\mathbf{E}}_{\mathrm{ext}}(t)\right), \quad (25) \end{aligned}$$

in agreement with the formula obtained in Mostepanenko (1979); and the result including the back-reaction, until order $\hbar$, is

$$\mathcal{E}(t) = \frac{1}{8\pi}|\mathbf{E}_{\text{ext}}(t)|^2 - \frac{7\alpha^2\left(\frac{\hbar}{mc}\right)^3}{5760\pi^2 mc^2}|\mathbf{E}_{\text{ext}}(t)|^4 + \frac{\alpha\left(\frac{\hbar}{mc^2}\right)^2}{960\pi^2}|\dot{\mathbf{E}}_{\text{ext}}(t)|^2. \tag{26}$$

For the average density of produced pairs, the result, until order $\hbar$, is

$$\begin{aligned}\mathcal{N}(t) = &\frac{\alpha}{512\pi mc^2}|\mathbf{E}_{\text{ext}}(t)+\mathbf{E}(t)|^2 + \frac{113\alpha^2\left(\frac{\hbar}{mc}\right)^3}{(64)^3\pi(mc^2)^2}|\mathbf{E}_{\text{ext}}(t)+\mathbf{E}(t)|^4 \\ &+ \frac{\alpha\left(\frac{\hbar}{mc^2}\right)^2}{3\pi(64)^2mc^2}\left(|\dot{\mathbf{E}}_{\text{ext}}(t)+\dot{\mathbf{E}}(t)|^2 - 2(\mathbf{E}_{\text{ext}}(t)+\mathbf{E}(t))\cdot(\ddot{\mathbf{E}}_{\text{ext}}(t)+\ddot{\mathbf{E}}(t))\right)\end{aligned} \tag{27}$$

Therefore, in the one-loop approximation we have

$$\begin{aligned}\mathcal{N}_0(t) := &\frac{\alpha}{512\pi mc^2}|\mathbf{E}_{\text{ext}}(t)|^2 + \frac{113\alpha^2\left(\frac{\hbar}{mc}\right)^3}{(64)^3\pi(mc^2)^2}|\mathbf{E}_{\text{ext}}(t)|^4 \\ &+ \frac{\alpha\left(\frac{\hbar}{mc^2}\right)^2}{3\pi(64)^2mc^2}\left(|\dot{\mathbf{E}}_{\text{ext}}(t)|^2 - 2\mathbf{E}_{\text{ext}}(t)\cdot\ddot{\mathbf{E}}_{\text{ext}}(t)\right).\end{aligned} \tag{28}$$

The result including the back-reaction, until order $\hbar$, is

$$\mathcal{N}(t) = \mathcal{N}_0(t) + \frac{\alpha}{256\pi mc^2}\mathbf{E}_{\text{ext}}(t)\cdot\mathbf{E}(t) + \frac{\alpha}{512\pi mc^2}|\mathbf{E}(t)|^2, \tag{29}$$

where we must take the induced electric field until order $\hbar^2$. However, since the induced electric field in the one-loop approximation has only terms at order $\hbar^{2n+1}$ with $n \in \mathbb{N}$, we can conclude that, at order $\hbar^2$, $\mathbf{E}(t)$ is equal to

$$\begin{aligned}\mathbf{E}_0(t) &+ \frac{\alpha\left(\frac{\hbar}{mc^2}\right)^2}{120\pi}\ddot{\mathbf{E}}_0(t) - \frac{7\alpha^2\left(\frac{\hbar}{mc}\right)^3}{360\pi mc^2}\mathbf{E}_0(t)|\mathbf{E}_{\text{ext}}(t)|^2 \\ &- \frac{7\alpha^2\left(\frac{\hbar}{mc}\right)^3}{180\pi mc^2}\mathbf{E}_{\text{ext}}(t)(\mathbf{E}_{\text{ext}}(t)\cdot\mathbf{E}_0(t)),\end{aligned} \tag{30}$$

where, $\mathbf{E}_0(t)$ is the induced electric field in the one-loop approximation until order $\hbar$ (see formula (14)).

Then, substituting (30) into (29), and taking only the terms until order $\hbar$, we obtain the density of produced pairs including the back-reaction.

5. THE BACK-REACTION PROBLEM FOR DISCONTINUOUS EXTERNAL FIELDS

From the results obtained in Section 2, we can prove that, for $\mathcal{C}^3$ external fields that are switched on and satisfy $\mathbf{E}_{\text{ext}}(t) \equiv \mathbf{0}\ \forall t > T$, the induced current density, until order $\hbar$, verifies

$$\mathbf{j}(t) = 0; \quad \forall t > T.$$

Here, we see that, when the external electric field has a discontinuity, it is not in the way shown above. In order to prove this, we take the potential $A^{\mu}_{\text{ext}}(\mathbf{x}, t) = (0, \mathbf{f}_{\text{ext}}(t))$, where we assume that $\mathbf{f}_{\text{ext}}(t)$ satisfies the following conditions:

$$\mathbf{f}_{\text{ext}}(t) = \mathbf{f}^{+}_{\text{ext}} \quad \text{for} \quad t > T; \qquad \mathbf{f}_{\text{ext}}(t) = \mathbf{f}^{-}_{\text{ext}} \quad \text{for} \quad t < -T.$$

$$\mathbf{f}_{\text{ext}} \in \mathcal{C}^{\infty}(\mathbb{R} \setminus \{T\}); \qquad \mathbf{f}_{\text{ext}} \in \mathcal{C}^{0}(\mathbb{R}); \quad \text{and} \quad \dot{\mathbf{f}}_{\text{ext}}(T^{-}) \neq \mathbf{0}.$$

Then, in the Heisenberg picture we have (see Fulling, 1989)

$$\begin{aligned} \hat{\psi}(\mathbf{x}, t) &= \sum_{\mathbf{k}\in\mathbb{Z}^3} \left(\hat{a}_{\mathbf{k},\text{in}} u_{\mathbf{k},\text{in}}(t) + \hat{b}^{\dagger}_{-\mathbf{k},\text{in}} u^{*}_{\mathbf{k},\text{in}}(t)\right) \frac{e^{\frac{2\pi i}{L}\mathbf{kx}}}{L^{3/2}} \\ &= \sum_{\mathbf{k}\in\mathbb{Z}^3} \left(\hat{a}_{\mathbf{k},\text{out}} u_{\mathbf{k},\text{out}}(t) + \hat{b}^{\dagger}_{-\mathbf{k},\text{out}} u^{*}_{\mathbf{k},\text{out}}(t)\right) \frac{e^{\frac{2\pi i}{L}\mathbf{kx}}}{L^{3/2}}, \end{aligned}$$

where the functions $u_{\mathbf{k},\text{in}}(t)$ and $u_{\mathbf{k},\text{out}}(t)$ are solutions of the equation $\hbar^2\ddot{u} + \epsilon^2_{\mathbf{k}}(t)u = 0$, and they have the asymptotic behavior

$$\lim_{t\to-\infty}\left[u_{\mathbf{k},\text{in}}(t) - \frac{1}{\sqrt{2\epsilon^{-}_{\mathbf{k}}}} e^{-\frac{i}{\hbar}\epsilon^{-}_{\mathbf{k}}(t+T)}\right] = 0;$$

$$\lim_{t\to\infty}\left[u_{\mathbf{k},\text{out}}(t) - \frac{1}{\sqrt{2\epsilon^{+}_{\mathbf{k}}}} e^{-\frac{i}{\hbar}\epsilon^{+}_{\mathbf{k}}(t-T)}\right] = 0,$$

with, $\epsilon^{\pm}_{\mathbf{k}} = \lim_{t\to\pm\infty} \epsilon_{\mathbf{k}}(t)$.

Note that, $\forall t \in \mathbb{R}$ we have $u_{\mathbf{k},\text{in}}(t) = A_{\mathbf{k}} u_{\mathbf{k},\text{out}}(t) + B_{\mathbf{k}} u^{*}_{\mathbf{k},\text{out}}(t)$. Therefore, the matching at the point $t = T$ provides

$$A_{\mathbf{k}} = \frac{W[u_{\mathbf{k},\text{in}}(T^{-}), u^{*}_{\mathbf{k},\text{out}}(T^{+})]}{W[u_{\mathbf{k},\text{out}}(T^{+}), u^{*}_{\mathbf{k},\text{out}}(T^{+})]}; \qquad B_{\mathbf{k}} = \frac{W[u_{\mathbf{k},\text{in}}(T^{-}), u_{\mathbf{k},\text{out}}(T^{+})]}{W[u^{*}_{\mathbf{k},\text{out}}(T^{+}), u_{\mathbf{k},\text{out}}(T^{+})]},$$

where $W[f, g] := f\dot{g} - \dot{f}g$ is the Wronskian of the functions f and g.

Now, to obtain the functions $u_{\mathbf{k},\text{in}}(t)$ and $u_{\mathbf{k},\text{out}}(t)$ we use the W.K.B. method (see Berry, 1982). We write

$$u_{\mathbf{k},\text{in}}(t) = \frac{1}{\sqrt{2\epsilon_{\mathbf{k}}(-T)}} e^{-\frac{i}{\hbar}\int_{-T}^{t} P_{\mathbf{k}}(s)ds}; \qquad u_{\mathbf{k},\text{out}}(t) = \frac{1}{\sqrt{2\epsilon_{\mathbf{k}}(T)}} e^{-\frac{i}{\hbar}\int_{T}^{t} P_{\mathbf{k}}(s)ds},$$

where $P_{\mathbf{k}}(t)$ satisfies the equation $-i\hbar\dot{P}_{\mathbf{k}}(t) = P_{\mathbf{k}}^2(t) - \epsilon_{\mathbf{k}}^2(t)$. We expand $P_{\mathbf{k}}(t)$ in power series of $\hbar$ thus, $P_{\mathbf{k}}(t) = \sum_{n=0}^{\infty} \hbar^n P_{n,\mathbf{k}}(t)$. We obtain, after having equalized the powers of $\hbar$

$$P_{0,\mathbf{k}}(t) = \epsilon_{\mathbf{k}}(t); \quad P_{1,\mathbf{k}}(t) = -\frac{i\dot{\epsilon}_{\mathbf{k}}(t)}{2\epsilon_{\mathbf{k}}(t)};$$

$$P_{2,\mathbf{k}}(t) = \frac{1}{2\epsilon_{\mathbf{k}}(t)}\left[-i\dot{P}_{1,\mathbf{k}}(t) - P_{1,\mathbf{k}}^2(t)\right];$$

$$P_{3,\mathbf{k}}(t) = \frac{1}{2\epsilon_{\mathbf{k}}(t)}\left[-i\dot{P}_{2,\mathbf{k}}(t) - 2P_{1,\mathbf{k}}(t)P_{2,\mathbf{k}}(t)\right];$$

$$P_{4,\mathbf{k}}(t) = \frac{1}{2\epsilon_{\mathbf{k}}(t)}\left[-i\dot{P}_{3,\mathbf{k}}(t) - 2P_{1,\mathbf{k}}(t)P_{3,\mathbf{k}}(t) - P_{2,\mathbf{k}}^2(t)\right]; \quad \text{etc.}$$

In order to obtain the Bogolubov coefficients, $A_{\mathbf{k}}$ and $B_{\mathbf{k}}$, we use the following approximation $P_{\mathbf{k}}(t) \sim \sum_{n=0}^{4} \hbar^n P_{n,\mathbf{k}}(t)$. The result is

$$A_{\mathbf{k}} = \frac{u_{\mathbf{k},\text{in}}(T)}{\sqrt{2\epsilon_{\mathbf{k}}(T)}}\left[2\epsilon_{\mathbf{k}}(T) + \sum_{n=1}^{4} \hbar^n P_{n,\mathbf{k}}(T^-)\right];$$

$$B_{\mathbf{k}} = -\frac{u_{\mathbf{k},\text{in}}(T)}{\sqrt{2\epsilon_{\mathbf{k}}(T)}} \sum_{n=1}^{4} \hbar^n P_{n,\mathbf{k}}(T^-).$$

Once we have obtained the Bogolubov coefficients, it is easy to check, for $t > T$, that

$$\begin{aligned}\langle 0|\mathcal{T}^\dagger(t)\hat{\mathbf{j}}(t)\mathcal{T}(t)|0\rangle &= \langle 0_{\text{in}}|\hat{\mathbf{j}}_{\text{out}}|0_{\text{in}}\rangle \\ &= -\frac{ec}{L^3}\sum_{\mathbf{k}\in\mathbb{Z}^3} \frac{c\mathbf{p}_{\mathbf{k}} + e\mathbf{f}_{\text{ext}}^{+}}{\epsilon_{\mathbf{k}}(T)}\left(2|B_{\mathbf{k}}|^2 + A_{\mathbf{k}}B_{\mathbf{k}}^* + A_{\mathbf{k}}^* B_{\mathbf{k}}\right),\end{aligned}$$

then, for $t > T$, an elementary but cumbersome calculation provides, until order $\hbar$, the following value of the induced current density

$$\begin{aligned}\mathbf{j}(t) = & -\frac{\alpha\left(\frac{\hbar}{mc^2}\right)^2}{480\pi^2}\ddot{\mathbf{E}}_{\text{ext}}(T^-) + \frac{7\alpha^2\left(\frac{\hbar}{mc}\right)^3}{1440\pi^2 mc^2}\left[\dot{\mathbf{E}}_{\text{ext}}(T^-)|\mathbf{E}_{\text{ext}}(T^-)|^2\right. \\ & \left. + 2\mathbf{E}_{\text{ext}}(T^-)\left(\dot{\mathbf{E}}_{\text{ext}}(T^-)\cdot\mathbf{E}_{\text{ext}}(T^-)\right)\right]. \end{aligned} \tag{31}$$

Remark 5.4. Note that, $\forall t > T$

$$\mathcal{N}(t) = \sum_{\mathbf{k}\in\mathbb{Z}^3} \langle 0_{\text{in}} | \hat{a}^\dagger_{\mathbf{k},\text{out}} \hat{a}_{\mathbf{k},\text{out}} | 0_{\text{in}} \rangle = \sum_{\mathbf{k}\in\mathbb{Z}^3} \langle 0_{\text{in}} | \hat{b}^\dagger_{\mathbf{k},\text{out}} \hat{b}_{\mathbf{k},\text{out}} | 0_{\text{in}} \rangle = \sum_{\mathbf{k}\in\mathbb{Z}^3} |B_{\mathbf{k}}|^2,$$

then, in the one-loop approximation, $\forall t > T$, we have

$$\begin{aligned} \mathcal{N}_0(t) = {} & \frac{\alpha}{512\pi mc^2} |\mathbf{E}_{\text{ext}}(T^-)|^2 + \frac{113\alpha^2 \left(\frac{\hbar}{mc}\right)^3}{(64)^3 \pi (mc^2)^2} |\mathbf{E}_{\text{ext}}(T^-)|^4 \\ & + \frac{\alpha \left(\frac{\hbar}{mc^2}\right)^2}{3\pi (64)^2 mc^2} \left(|\dot{\mathbf{E}}_{\text{ext}}(T^-)|^2 - 2\mathbf{E}_{\text{ext}}(t) \cdot \ddot{\mathbf{E}}_{\text{ext}}(T^-) \right), \end{aligned} \tag{32}$$

and consequently, the average density of produced pairs does not converge to zero when $t \to \infty$. This is a consequence of the discontinuity of the external field.

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